Reference: The Algebra of Functions

Domains of real-valued functions

- The **domain** of a function is the set of all of its allowed inputs ("x" values).
- Sometimes a function will come with a restricted domain – like " $f(x)=10+x$ for $x>2$ ". This function's domain only allows x's greater than 2. Why? Who knows. That's just how it is.
- Functions which are **real-valued** – that is, they always give a real number as their answer – always have certain domain restrictions. The restrictions usually won't be spelled out for you explicitly. You have to be able to recognize them based on these rules:
  - Rule #1: Don't divide by zero.
  - Rule #2: Don't take an even-indexed root of a negative number.
  - Rule #3: Don't take a logarithm of zero or a negative number.
- Example: $f(x)=\dfrac{\sqrt{x+1}}{x-3}$.
  - Rule #1: This function includes division, so I need to check this. Is it possible for me to plug in an x that would make the denominator, x-3, be zero?
    - $x-3=0\,?\;\rightarrow\; x=3$

    Yes! x-3 could be zero if x=3. So, I have to take "3" out of the domain for the function.
  - Rule #2: This function has a root with an index of 2 (even), so I need to check this. Is it possible to plug in an x that would give me a negative number for the radicand, $x+1$?
    - $x+1<0\,?\;\rightarrow\; x<-1$

    Yes! If x is less than -1, the radicand would be negative. -1 itself is fine... that would make the radicand 0. But, I need to throw out all x's that are less than -1.
  - The domain is therefore:
    - All values of x except for x=3 and x<-1, because those would break rules. Or, you could say " $x\neq 3$ and $x\geq -1$ ". Or in interval notation, that would be " $[-1,3)\cup(3,\infty)$ ".

Function Operations

- You can combine functions using any of the basic arithmetic operations: $+$, $-$, $\times$, and $\div$. These combined functions are often written in a condensed style as shown below:
  - $f(x)+g(x)=(f+g)(x)$
  - $f(x)-g(x)=(f-g)(x)$
  - $f(x)\cdot g(x)=(f\cdot g)(x)$
  - $f(x)\div g(x)=(f\div g)(x)$
- The domain for the combined function has all the restrictions that the originals did, plus you need to check the new formula to see if any new problems have come up.
- Example: $f(x)=x-3, g(x)=x^2-9$. These functions have no domain problems on their own! All values of x are OK. But if I divide them, something happens...
  - $\left(\dfrac{f}{g}\right)(x)=\dfrac{x-3}{x^2-9}$. Before simplifying at all, I can see that there are two values of x which could mess things up: 3 and -3, because they would both make the denominator zero.
  - Now I can simplify, taking the domain along in each step:
  - $\left(\dfrac{f}{g}\right)(x)=\dfrac{x-3}{x^2-9}\text{ for } x\neq 3,-3 \;\rightarrow\; \dfrac{x-3}{(x-3)(x+3)}\text{ for } x\neq 3,-3 \;\rightarrow\; \dfrac{1}{(x+3)}\text{ for } x\neq 3,-3$
  - If I had looked only at the simplified form, I wouldn't know that +3 needed to be removed from the domain! That's why it's important to check the domain before simplifying!

Composition of Functions

- If you plug one entire function into another, it's called a **composition of functions**.
- Compositions can be written in two styles, just like operations: $f(\ g(x)\ ) = (f \circ g)(x)$. In the second style, that's a little circle between them... not a dot, letter "o", or zero. A circle.
- When evaluating a composition, you do the inside function first!
- Example: Suppose $f(x) = x+10$ and $g(x) = 5 \cdot x$.
  - $f(g(3)) =$? Well, $g(3) = 15$. So... $f(\ g(3)\ ) = f(\ 15\ ) = 15+10 = 25$.
  - Even without a number, you can plug one into the other: $f(\ g(x)\ ) = f(\ 5x\ ) = 5x+10$.
  - Just remember, you need to replace each "*x*" in the formula with whatever is in the parentheses (called the "argument"):
    $f(15) = 15+10$, $f(z) = z+10$, $f(\pi) = \pi+10$, $f(\text{your mom}) = \text{your mom}+10$

Inverse Functions

- For a function *f*(*x*), if you swap the domain and the range, you create a new relation called the **inverse** of *f*. The inverse is written $f^{-1}(x)$.
  - The inverse might be a function too, but sometimes it's just a regular relation.
  - The graphs of *f* and f^{-1} are reflections of each other, as if you put a mirror down along a line at 45° and looked at the image inside.
  - The domain of *f* becomes the range of f^{-1}, and vice-versa.
- The formula or rule for an inverse can be found by following these steps...
  - Starting with the function $\boxed{f(x) = 6x+10}$...
  - Rewrite the function using a "y" instead of the function notation: $y = 6x+10$
  - Create the inverse by changing every "x" to "y" and vice-versa: $x = 6y+10$
  - Solve the new equation for "y". This will normally take a few steps!!! $y = \frac{1}{6}x - \frac{5}{3}$.
  - Rewrite in function notation by taking away the "y": $\boxed{f^{-1}(x) = \frac{1}{6}x - \frac{5}{3}}$
- When you compose a function and its inverse, they will (usually) cancel each other out. After all, they are opposites! Using the example *f* and f^{-1} from just above...
  - $f(\ f^{-1}(12)\ ) = f(\ \frac{1}{6} \cdot 12 - \frac{5}{3}\ ) = f(\ 2 - \frac{5}{3}\ ) = f(\ \frac{1}{3}\ ) = 6 \cdot \frac{1}{3} + 10 = 2+10 = 12$
  - Without all the steps in between, it's just: $f(\ f^{-1}(12)\ ) = 12$.
- Composing a function and its inverse doesn't always work, though! If one or both of them have a restricted domain, you can run into problems. For example:
  - Consider this function and its inverse: $g(x) = x^2+3$, $g^{-1}(x) = \pm\sqrt{x-3}$.
  - If we use a number in both of their domains, like 12, all is well:
    $g(g^{-1}(12)) = g(\ \pm\sqrt{12-3}\ ) = g(\ \pm\sqrt{9}\ ) = g(\pm 3) = (\pm 3)^2+3 = 9+3 = 12$
  - But if I use a number that doesn't belong to both domains, like -12...
    $g(g^{-1}(-12)) = g(\ \pm\sqrt{-12-3}\ ) = g(\ \pm\sqrt{-15}\ ) = ???$
  - The square root of -15 doesn't exist – it would be imaginary. So I can't actually evaluate this. The functions can only cancel each other out if both of them work!
  - Even when you CAN evaluate the composition, sometimes you'll get two answers. For example, consider $h(x) = (x-8)^2$, $h^{-1}(x) = \pm\sqrt{x}+8$:
    $h^{-1}(\ h(2)\ ) = h^{-1}(\ (2-8)^2\ ) = h^{-1}(36) = \pm\sqrt{36}+8 = \pm 6+8 = 12$ or 2 ... instead of 2 alone!