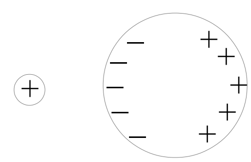
ELECTRICITY!

charges - qualitatively

- After certain types of objects are rubbed against each other, they begin exerting forces on each other and other objects, even without coming into physical contact. This simple observation is the starting point for all the physics of electricity.
- Experimentation with a variety of materials indicates that there are two types of charge.
    - Like charges repel each other. Opposite charges attract each other.
    - Either type of charge will attract a neutral object because of polarization (see below).
- Locations of charge density:
    - Conductors (in equilibrium) can be charged only on their surfaces. If some charge *were* to be concentrated somewhere within a conductor's interior, the electric repulsion between the charges would cause them to move and spread out as far as possible – right to the surface.
    - Charges on a conductor's surface are *more concentrated* at protruding edges and points than smooth parts and indentations. "Smaller" or tighter curves have larger charge densities.
    - Insulators, on the other hand, can hold a charge concentration anywhere because the charges are not free to move around. Deposit charge anyplace you want: their surface, inside, whatever.
- When a charged surface touches another surface, charges will often transfer from one to the other.
    - Conductors that are brought into contact will equalize their electric potentials.
        - A small sphere will have higher charge density than a big sphere, but still less total charge.
        - If the objects are identical, their *total charge* equalizes too.
    - Even insulators that are brought into contact will trade excess charge at their point of contact.
- In the presence of an external charge, neutral objects become *polarized*.
    - In conductors, charges are free to move anywhere, so they respond to the forces they feel from the external charge, moving towards or away from it.
    - In insulators, although charges can't leave their "home" atoms, an external charge can distort the shapes of atoms and molecules.
    - The result in both cases is a concentration of opposite charges on the near side, and an equal concentration of like charge on the far side.
    - The interior of a polarized object is unaffected – only its surface is charged.
    - The attracting charges are closer together than the repelling charges, so the net electrical force between an external charge and a polarized object is attractive.



- Grounding an object puts it into electrical contact with the Earth, an infinite reservoir of charges.
    - An *isolated*, grounded conductor will neutralize because any excess charges on it are now free to spread out across the entire planet.
    - A *grounded* conductor in proximity to an external charge will *acquire* a net charge by drawing charges out of the Earth that are opposite in sign to the external charge.
- The behavior of charges can be conceptualized in four ways:
    - Charges directly exert forces on other charges and move accordingly.
    - Charges' interactions with each other form potential energy; they move to minimize this energy.
    - Charges create electric fields and can feel the electric fields of other charges. Electric fields exert forces on charges, which then move accordingly.
    - Charges create electric potential and can feel the electric potential of other charges. Depending on their sign, charges move towards areas of either higher or lower electric potential.
- These four ideas are equally valid and interchangeable. However, electric fields are usually treated as the "real" or most fundamental conceptualization.

charges - quantitatively

- The total amount of charge is always conserved in any process. $Q_{initial} = Q_{final}$.
- Electric force, electric field, electric potential, and potential energy are closely related concepts and it is no coincidence that their formulae are so similar:

	Scalers	***Vectors***	
Intrinsic A single charge has this. Value found at a point in space.	$V=\frac{1}{4\pi\epsilon_0}\frac{q}{r}$ *Electric Potential*	$\vec{E}=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ *Electric Field*	↓ Multiply by a second charge
Interaction A pair of charges has this. Value found for the system.	$U=\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$ *Potential Energy*	$\vec{F}=\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$ *Electric Force*	
	→ Take spatial derivatives (-*d*/*dx*, -*d*/*dy*, -*d*/*dz*)		

- The four formulae above apply *only to point charges*, and not all electric fields in general.
- However, the "multiply by *q*" and "take a derivative" relationships apply broadly:
  - For a charge *q* at any point in any electric field / electric potential, the following are true:

$$\vec{F}=q\vec{E} \qquad \Delta U=q\,\Delta V$$

  - The derivative relationships are also true for all electric fields, and work like this:

$$\vec{E}=-\frac{dV}{d\vec{r}}, \quad \text{or} \quad E_x=-\frac{dV}{dx}, E_y=-\frac{dV}{dy}, \text{ and } E_z=-\frac{dV}{dz}$$

$$\vec{F}=-\frac{dU}{d\vec{r}}, \quad \text{or} \quad F_x=-\frac{dU}{dx}, F_y=-\frac{dU}{dy}, \text{ and } F_z=-\frac{dU}{dz}$$

  - If that electric field is uniform (such as the field between parallel charged plates), and Δd is a displacement that is measured along field lines (not across them), the derivatives simplify:

$$\Delta U=F\,\Delta d \qquad \Delta V=E\,\Delta d$$

  - In the first, Δd is a displacement through which a charged particle has been moved (there can be no potential energy *U* without a charge there). In the second, Δd is a distance between two points in space at which the potential is measured (because there needn't be any charge present for a potential or field to exist). Displacements perpendicular to the field have no ΔU or ΔV.
- The four main formulae in the table above can be adapted to apply to any charge distribution instead of only point charges. To do so, you must apply superposition (i.e., "adding stuff up").
  - For discrete charges, simply calculate the quantity you want with respect to every charge *q* and add them together. (If adding vectors, don't neglect their directions and components.)
  - For a continuous "smear" instead of a collection of point charges, do an integral over all charge elements d*q* instead of a plain sum. (Again, being careful about vectors.)
- Setting up integrals over charge distributions:
  - Choose a charge element d*q* at some arbitrary location within the distribution.
  - Write d*q* in terms of some integrable coordinate (x, y, z, r, θ, ϕ) and a charge density (ex. λ dx).
  - Write *r* in terms of the same coordinates so its value will change while you integrate!
  - If integrating a vector quantity (force or field) split your work into an integral for each component by multiplying by the appropriate trig functions.
  - Set the limits on your integral – make sure that the location of "zero" indicated by your limits (that is, the origin for your coordinates) is the same as the zero indicated by your definition of *r*. It is a fairly common error for these two origins to disagree.

Gauss' law

- This law provides a way to calculate electric fields that is significantly easier than Coulomb.

$$\Phi_{\mathrm{E}} = \oint \vec{\boldsymbol{E}} \cdot \mathrm{d}\vec{\boldsymbol{a}} = \frac{q_{\mathrm{enc}}}{\epsilon_0}$$

- Over any closed surface, called a Gaussian surface, integrate $\vec{\boldsymbol{E}}$ dotted with elements of the surface's area. The result is proportional to the total charge contained in the space bound by that surface. It is also referred to as "electric flux". (See the section on flux near the end of this packet.)
    - You can calculate an electric flux through any surface, even an open one. However, Gauss' Law – which relates that flux to the charge – only works if the surface is closed. That's why there's a circle on the integral sign, to remind you!
- The usefulness of this law is not immediately apparent. After all, it appears to require that you already know $\vec{\boldsymbol{E}}$ in order to do the integral.
    - *The power of Gauss' Law lies entirely in the choice of an appropriate surface.*
    - Over any patch of the surface where $\vec{\boldsymbol{E}}$ and $\mathrm{d}\vec{\boldsymbol{a}}$ are perpendicular, the integrand is zero, and thus the integral over that patch is zero.
    - Over any patch of the surface where $\vec{\boldsymbol{E}}$ and $\mathrm{d}\vec{\boldsymbol{a}}$ are parallel or antiparallel, the integrand becomes $\pm E\,\mathrm{d}a$. Further, if the electric field strength is constant over this patch of the surface, $\pm E$ can be taken out front and the integral over that patch is just its area.
    - (If you ever want to calculate the flux, it's easy – just $q_{\mathrm{enc}}/\varepsilon_0$ – but it doesn't really do you any good unless you were lucky enough to be asked for it instead of $\vec{\boldsymbol{E}}$.)
- On a good Gaussian surface, one of the following is true at every point on the surface:
    - $\vec{\boldsymbol{E}}$ and $\mathrm{d}\vec{\boldsymbol{a}}$ are perpendicular.
    - $\vec{\boldsymbol{E}}$ and $\mathrm{d}\vec{\boldsymbol{a}}$ are parallel (or antiparallel) and the magnitude, E, is unchanging.
    - $\vec{\boldsymbol{E}}$ is zero.
- Using such a surface, split the full surface integral into integrals for each patch based on which shortcut applies, and add the results together.
- In this way, you can evaluate the integral without knowing anything about $\vec{\boldsymbol{E}}$ aside from it being constant in strength over part of your surface. $\vec{\boldsymbol{E}}$ comes out of the integral, and then you can solve for it afterwards.
- Be warned that Gauss' Law is not magic! It will not solve every problem. It is only useful in cases where you can construct an imaginary surface that meets the criteria above.

CIRCUITS!

capacitors

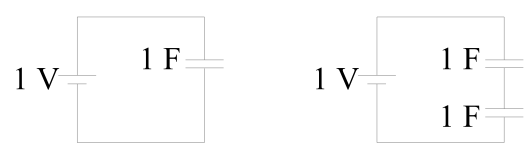
- The intrinsic property of a capacitor is its capacitance, which we define in a couple of ways:

$$C=\frac{Q}{V}=\frac{\epsilon_0 A}{d}$$

- Although this is mathematically simple, its conceptual meaning is not immediately obvious.
  - C is a measurement of how much charge you need to push into the capacitor in order to get a Volt of potential difference across it.
  - For each amount of charge, Q, that you put in, the voltage across the capacitor goes up by one.
  - A "small" capacitor with a small value of C can reach a large voltage with only a small amount of charge. A capacitor with a larger C needs a lot more charge to reach the same voltage.
- Insulating materials placed between the plates of a capacitor are called *dielectrics*.
  - Capacitors are usually assumed to have a vacuum (or air) between their plates.
  - However, the presence of a dielectric will increase the capacitance.
  - This is because the dielectric itself becomes polarized, reducing the electric field between the plates and therefore lowering the voltage reached at a given charge, Q.
  - With a dielectric in place, more charge is needed to reach a given voltage.
  - A dielectric increases C by a factor called its dielectric constant, represented by κ. If the capacitance when filled with vacuum is C_0, then the dielectric-filled capacitance is:

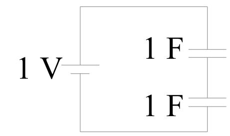
$$C=\kappa\, C_0=\kappa\frac{\epsilon_0 A}{d}$$

- When capacitors are arranged in parallel...
  - They all have the same potential difference.
  - Their capacitances add directly. More capacitors means *more* capacitance.
    - Imagine side-by-side capacitors as being a single capacitor with a larger area. C is proportional to area, so, more area means more capacitance.
- When capacitors are arranged in series...
  - They all have the same current.
  - They all have the same charge. (Usually...)
    - Normally, all elements in a circuit start out being electrically neutral (no net charge).
    - If two or more neutral capacitors are in series, the "interior" plates will have equal and opposite charges. This is a result of charge conservation.
    - However, *if extra charge is present* between the capacitors, they will end up with different charges. For more detail, go down a couple bullets...
- Their capacitances add by reciprocals. More capacitors means *less* capacitance. Why?
    - Compare the following circuits...
    - In the first, the battery has to supply 1 C of charge before the voltage of the capacitor reaches 1 V and the current stops: $V = Q / C = 1\text{ C} / 1\text{ F} = 1\text{ V}$.



    - In the second, the battery only has to supply 0.5 C of charge. When the top capacitor is charged to 0.5 C, the bottom capacitor is automatically charged to the same amount. Each capacitor then has $V = 0.5\text{ C} / 1\text{ F} = 0.5\text{ V}$ across it. That gives a total of 1 V. So, if we were to replace the pair with a single capacitor, it would have to be a capacitor which reaches 1 V when only 0.5 C are on it. In other words, a 0.5 F capacitor.

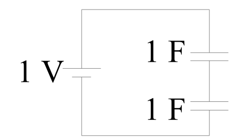
- Capacitors store energy.
  - When a capacitor is partially charged, it takes *work* to add more charge. Each extra electron placed on the negative plate needs to be pushed into place against an electrical repulsion.



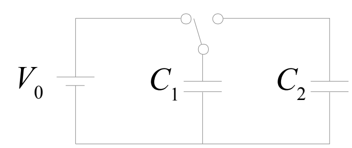
  - When discharging, a capacitor can push charges to generate a current much like a battery does. In this way, the energy put into the capacitor by charging it can be recovered later.
  - The energy stored can be written three ways if we use $Q = V C$ to change things around. We use the symbol U for this energy since it's an electric potential energy:

$$\text{Energy} = U = \frac{Q^2}{2C} = \frac{1}{2} Q V = \frac{1}{2} C V^2$$

- Explaining series capacitors with different charges:
  - This is an aside. It takes a lot of space to explain, but is rarely relevant.
  - In the circuit to the right, each cap will charge to 0.5 C, and $Q_2 - Q_1 = 0$.



    - If we deposit +0.25 C onto the area between the two capacitors, then $Q_2 - Q_1$ has to be 0.25 C.
    - Additionally, once the circuit has reached a new equilibrium, $V_1 + V_2$ still needs to be 1 V.
    - A little algebra (try it yourself) dictates that the new charges are 0.375 C and 0.625 C. So, even though the capacitors are in series with each other, they do not have the same charge.
  - The most common way for “extra” charge to factor in is when an already-charged capacitor is placed into a circuit, or a switch is thrown which effectively does the same thing. For example:
    - When the switch is to the left, Capacitor 1 will charge.



    - When the switch is thrown to the right, the two capacitors can be thought of as in series with each other. (Or as in parallel – when there are only two elements in a circuit, the two descriptions are equally valid.)
    - However, because of the charge that was residing on the plates of Capacitor 1, the top half of the circuit will have a net positive charge, and the bottom half a net negative charge.
    - Q_1 and Q_2 will be equal in this case only if the capacitances are the same.

resistors

- The intrinsic property of a resistor is its resistance, which we define in a couple of ways:

$$R = \frac{V}{I} = \frac{\rho L}{A}$$

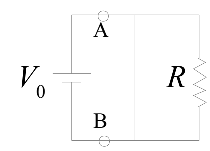
- When resistors are arranged in series...
  - They all have the same current.
  - Their resistances add directly. More resistors means *more* resistance.
    - Imagine end-to-end resistors as being a single resistor with a larger length. R is proportional to length, so, more length means more resistance.
- When resistors are arranged in parallel...
  - They all have the same voltage.
  - Their resistances add like reciprocals. More resistors means *less* resistance.
    - Imagine side-by-side resistors as being a single resistor with a larger area. R is inversely proportional to area, so, more area means less resistance.

- Resistors dissipate energy.
  - When current encounters a resistor, it takes *work* to "push" the charges through.
  - However, the energy put into a resistor in this fashion cannot be recovered later like the energy put into a capacitor. This energy is continuously dissipated as heat at a rate that depends on the current through (or voltage across) that resistor.
  - The power dissipation can be written three ways if we use $V = I\,R$ to change things around:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = I^2 R = I\,V = \frac{V^2}{R}$$

(ideal) batteries

- Batteries are NOT sources of constant current.
  - The current that courses through a battery depends on the entire circuit, not just the battery.
- Batteries are NOT charged on their ends.
  - We label them + and − but those labels only tell us which end has a higher electric potential.
- Ideal batteries ALWAYS have a potential difference of V_{battery} between their terminals.
  - That's basically the definition of an ideal battery – it maintains that voltage no matter what.
  - This definition can lead to conflicts with our usual assumption that wires have zero resistance.
  - In this circuit the battery has been shorted. What is the potential difference between A and B?
    - If we look at the battery, it appears that $V_{AB} = V_0$.
    - If we look at the shorting wire, it appears that $V_{AB} = 0$.
    - These cannot both be true! The problem is in our model of the circuit. In a case like this you can't assume both the wires and battery are ideal.
    - If you run into a circuit like this and feel confused, don't worry. When assuming ideal circuit elements, it *is* confusing.



RC circuits

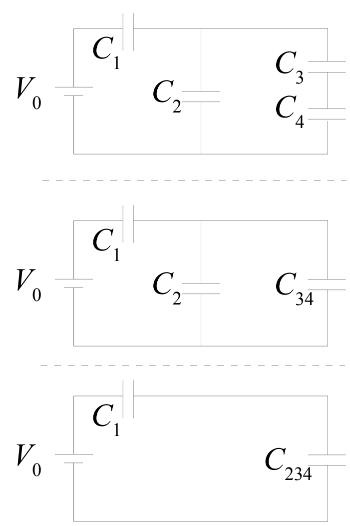
- The charging and discharging of capacitors are exponential functions – they start fast and end slow. Why is this the case?
  - As an empty capacitor fills up with charges, it gets progressively harder to shove more charge onto the plates. The first few charges are easy, but as the plate gets more and more charged, it gets more and more difficult to overcome the charges' mutual repulsion.
  - Similarly, when discharging, the first few charges to leave the plate are feeling a strong electric field. As the plate loses charge, the repulsion becomes weaker and the rate, again, drops.
- There are two forms for the equations that describe charging and discharging capacitors:
  - Since the charge buildup and voltage are proportional to each other, the equations that describe them have the same form.
    - When charging, a rapid increase that levels off.
    - When discharging, a rapid decline that levels off.
  - The current, however, always declines. For the current entering a capacitor to increase over time, it would have to be getting *easier* to put more charges on the plates!

$$\text{Discharging: } \frac{Q}{Q_0} = \frac{V}{V_0} = \frac{I}{I_0} = \mathrm{e}^{(-t/\tau)} \qquad \text{Charging: } \frac{Q}{Q_0} = \frac{V}{V_0} = \left(1 - \mathrm{e}^{(-t/\tau)}\right),\ \frac{I}{I_0} = \mathrm{e}^{(-t/\tau)}$$

  - The time constant, τ, has dimensions of time and is equal to $\tau = R\,C$.
  - Current is the same equation both ways because, as you may recall, current is defined as $\mathrm{d}q/\mathrm{d}t$. And, ignoring signs in front (which just tells you the direction of the current), the derivatives of both Q equations are the same. (Try it!)

circuit analysis

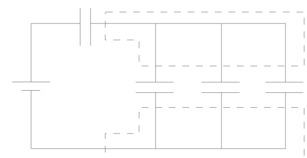
- When reducing series or parallel resistors or capacitors, how do you know which elements will be the most important?
  - In directly-added arrangements, the “largest” elements dominate.
  - In reciprocally-added arrangements, the “smallest” elements dominate.
    - If you have a very low-resistance resistor in place, it is already easy for current to flow. Adding additional pathways with parallel resistors will help, but what's most important is that you have that original low-resistance path. Adding alternate high-resistance paths makes only a small difference.
    - If you have a very low-capacitance capacitor in place, it already needs just a very small charge to reach your full voltage. Adding more capacitors in series will help you reach full voltage quicker, but what's most important is that you have that original capacitor which fills quickly. Adding high-capacitance elements in series makes only a small difference.
  - For example, a series of 10 kΩ, 100 Ω, and 1 Ω will be closest to the largest resistor: 10 kΩ. A parallel combination of the same resistors will be closest to the smallest: 1 Ω.
- Voltage across a resistor is a function of the current through that resistor.
- Voltage across a capacitor is a function of the charge buildup on that capacitor. This charge buildup is a function of time. Usually, we only care about the “fully charged” voltage.
- Series and parallel rules that apply to all types of circuit element:
  - Elements in series all have the same current.
  - Elements in parallel all have the same voltage.
- Determining whether elements are in series or in parallel is not always easy. It is very common for two elements to be *neither* in series *nor* in parallel with each other.
  - For two elements to be in series, there must be no choice but to go through both of them if you are going to go through one. There must be no nodes in between the two elements.
  - For two elements to be in parallel, they must be the only elements on their respective paths. If either path has more than one element on it, the elements are not in parallel.
- Consider the circuit to the right:



  - Capacitor *1 is not in series* with any other capacitor. There is a node between it and the other capacitors.
  - Capacitor *2 is not in parallel* with any other capacitor. The other path has two elements on it.
  - However, capacitors *3 and 4 are in series* with each other. There are no nodes between.
  - If we replace 3 and 4 with a single equivalent capacitor, C_{34}, then we can say that *2 is parallel to 34*. Both 2 and 34 are the only elements on their paths.
  - If we replace 2 and 34 with a single equivalent capacitor, C_{234}, then we can say that *1 is in series with 234*. There are no nodes between them.
  - Upon combining 1 with 234, the circuit is reduced to just a battery and one capacitor.

Kirchhoff's laws

- First step: reduce the circuit as far as possible by finding equivalent capacitors and resistors.
  - Many circuits cannot be fully reduced. Some cannot be reduced at all. To reduce a circuit you must find resistors or capacitors that are in series or parallel with each other.
- Second: look for shortcuts. Sometimes you can tell what the voltage for an element will be simply by examining the circuit diagram.
  - For example, any element that is connected directly across a battery will have that battery's voltage across it. There is no need to do a loop rule to calculate this.
  - In some circuits, after doing a reduction and looking for shortcuts you might be done without actually applying Kirchhoff's Laws directly.
- Third: assign a current name and direction to each branch in your (reduced) circuit.
  - You should have one current for every node-to-node pathway in the whole circuit.
  - It doesn't matter what directions you pick for the currents. If you guess wrong, you will merely have a minus sign in front of the current when you find its value.
  - Remember that a node is not just a point in the diagram where lines intersect, but the entire stretch of wire between circuit elements. In the circuit shown here, there are only two nodes, indicated by the dashed lines. Each node has four wires coming out of it.
    - This circuit can easily be reduced, but if you were to solve it using Kirchhoff Laws without reducing it, you would need four currents.



- Fourth: Apply the *Node Rule*. At every node, the amount of current coming in must equal the amount of current going out: $I_{in} = I_{out}$.
  - You will not need to write the node rule for every node in the circuit. Only until every named current has appeared in at least one equation.
- Fifth: Apply the *Loop Rule*. Pick any "loop" in the circuit that gets you from some starting place through at least two circuit elements and back to your starting point. For each element that your loop passes through, add or subtract the voltage across that element. When you get back to your starting point, write "= 0".
  - Whether you add or subtract the voltage depends on the direction of your loop.
  - For batteries, if your loop takes you from...
    - ...the − terminal to the + terminal, add $V_{battery}$.
    - ...the + terminal to the − terminal, subtact $V_{battery}$.
  - For resistors, if your loop takes you through the resistor...
    - ...in the same direction as that resistor's current, subtract $I R$.
    - ...in the opposite direction as that resistor's current, add $I R$.
  - For capacitors, if your loop takes you from...
    - ...the − side to the + side, add Q / C.
    - ...the + side to the − side, subtract Q / C.
  - You will not need to write the loop rule for every loop in the circuit. There is always at least one redundant loop, but it is not always easy to tell how many you will need.
- Once you have enough equations (at least as many as unknown currents), solve for the currents using algebra or a matrix.

MAGNETISM!

magnetic fields - qualitatively

- Discovering magnetism:
  - We discovered electricity because we noticed that after rubbing certain objects together they started exerting forces on other objects.
  - Similarly, we discovered magnetism because we noticed that some objects made of certain materials "naturally" exert forces on similar objects.
  - Objects made from these materials that don't naturally exert forces can be "trained" to do so.
  - The forces appear to mainly affect electrical conductors.
  - Because these forces act between objects that do not touch each other, we can theorize a magnetic field to describe the forces which is analogous to an electric field.
- Electric fields and magnetic fields are both 3D vector fields, but they have important differences:

Electricity	***Magnetism***
Field is produced by the existence of charges.	Field is produced by the motion of charges.
Field lines always start and end on charges.	Field lines are always loops with no ends.*
Forces are along field lines.	Forces are perpendicular to field lines.
Forces affect all charges.	Forces affect charges moving w.r.t. field lines.**

  - * Some diagrams of bar magnets may make it appear that magnetic field lines start on North poles and end on South poles. However, the field lines actually penetrate through the magnet and connect together in the middle, forming loops.
  - ** That is, charges moving through a static field *or* static charges in a changing field.
- A compass needle will rotate to align itself with a magnetic field, and can thus be used to trace field lines. Our knowledge of the shapes of magnetic fields comes from such measurements.

magnetic fields - quantitatively

- A single moving charge creates a field as described by the Biot-Savart law: $\vec{\boldsymbol{B}}=\frac{\mu_0}{4\pi}\frac{q\,\vec{\boldsymbol{v}}\times\hat{\boldsymbol{r}}}{r^2}$
- The field lines for a single moving charge are concentric circles lying in planes perpendicular to the charge's velocity. The field is strongest close to the charge.
- Electric current is the motion of a large charge distribution. If we consider a small piece of the distribution with a charge $\mathrm{d}q$, it will create a small contribution to the total magnetic field $\mathrm{d}\vec{\boldsymbol{B}}$. We can then convert Biot-Savart from a rule about one moving charge to a rule about a full current.

$$\mathrm{d}\vec{\boldsymbol{B}}=\frac{\mu_0}{4\pi}\frac{\mathrm{d}q\,\vec{\boldsymbol{v}}\times\hat{\boldsymbol{r}}}{r^2}$$

If $\vec{\boldsymbol{s}}$ is the position of our charge $\mathrm{d}q$, and $\vec{\boldsymbol{r}}$ is the vector between $\mathrm{d}q$ and our point of interest:

$$\mathrm{d}q\,\vec{\boldsymbol{v}}=\mathrm{d}q\left(\frac{\mathrm{d}\vec{\boldsymbol{s}}}{\mathrm{d}t}\right)=\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)\mathrm{d}\vec{\boldsymbol{s}}=I\,\mathrm{d}\vec{\boldsymbol{s}} \quad\rightarrow\quad \mathrm{d}\vec{\boldsymbol{B}}=\frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{\boldsymbol{s}}\times\hat{\boldsymbol{r}}}{r^2}$$

- Biot-Savart Law : magnetism :: Coulomb's Law : electricity
  - Biot-Savart can be used, in principle, to determine the magnetic field caused by *any* combination of moving charges / currents.
  - However, doing so may be a big pain in the butt depending on how they are arranged.
- The field lines for a single straight (non-curving) current are concentric circles lying in planes perpendicular to the current. The field strength diminishes with distance from the current, but is uniform with respect to motion along the current.

Ampere's law

- This law provides a way to calculate magnetic fields that is significantly easier than Biot-Savart.

$$\oint \vec{B} \cdot \mathrm{d}\vec{s} = \mu_0 I_{\text{enc}}$$

- Along any closed path, called an Amperian loop, integrate $\vec{B}$ dotted with elements of the path. The result is proportional to the total current penetrating a surface bound by that path.
- The usefulness of this law is not immediately apparent. After all, it appears to require that you already know $\vec{B}$ in order to do the integral.
  - *The power of Ampere's Law lies entirely in the choice of an appropriate path.*
  - Along any segment of the path where $\vec{B}$ and $\mathrm{d}\vec{s}$ are perpendicular, the integrand is zero, and thus the integral along that segment is zero.
  - Along any segment of the path where $\vec{B}$ and $\mathrm{d}\vec{s}$ are parallel or antiparallel, the integrand becomes $\pm B\ \mathrm{d}s$. Further, if the magnetic field strength is constant along this segment of the path, $\pm B$ can be taken out front and the integral along that segment is just its path length.
- In a good Amperian loop, one of the following is true at every point on the path:
  - $\vec{B}$ and $\mathrm{d}\vec{s}$ are perpendicular.
  - $\vec{B}$ and $\mathrm{d}\vec{s}$ are parallel (or antiparallel) and the magnitude, B, is unchanging.
  - $\vec{B}$ is zero.
- Using such a loop, split the full path integral into integrals for each segment based on which shortcut applies, and add the results together.
- In this way, you can evaluate the integral without knowing anything about $\vec{B}$ aside from it being constant in strength along part of your loop. $\vec{B}$ comes out of the integral, and then you can solve for it afterwards.
- Be warned! Like Gauss' Law, Ampere is not magic! It is only useful in cases where you can construct an imaginary loop that meets the criteria above.

magnetic forces

- Unlike electric fields, which exert forces on all charged objects, magnetic fields only affect charges that are in motion relative to the field.
  - The fundamental relation describing the magnetic force on a single moving charge is:

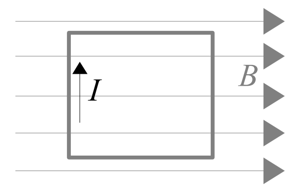
$$\vec{F} = q\vec{v} \times \vec{B}$$

  - In an electrical current there are many moving charges. Suppose a wire carrying a current, I, has a length L. Assuming the charges move with a constant velocity, we can write $\vec{v}$ in terms of L and the time t that they take to traverse the wire. This makes the magnetic force on a wire...

$$\vec{F} = q\frac{\vec{L}}{t} \times \vec{B} = \frac{q}{t}\vec{L} \times \vec{B} = I\,\vec{L} \times \vec{B}$$

  - Here the vector $\vec{L}$ points along the current and has magnitude equal to the length of the wire.
  - Note the cross product in these force equations: Not *all* motion is equally important. The larger the angle between the motion and the field vectors, the stronger the force is.
    - Charges are free to move *along* magnetic field lines with no interference.
    - The force is *strongest* when the charge's velocity is *perpendicular* to the field.

- Even an object which has no net electrical charge can feel a magnetic force depending on the motion of the positive and negative charges within it.
  - For example, a neutral wire carrying an electric current has charges of one sign moving through it while charges of the other sign remain motionless.
  - A permanent magnet has its atoms aligned in such a way that the "spins" of their charged nuclei and the spins and motion of their electrons are (on average) pointing in the same direction.
- As written, this formula applies to straight segments of wire and uniform magnetic fields. However, if the strength of $\vec{B}$ or the angle between $\vec{B}$ and $\vec{L}$ varies from place to place on the wire, the wire can be split into segments and force on each segment found independently. For example:

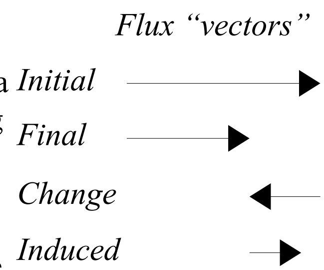


  - Consider the current-carrying loop shown here. To find the total force on the loop, find the force on each side and then add the four force vectors together.
  - If the variation is continuous instead of discrete, break things up with an integral instead of a simple sum.

INDUCTION!

induced currents

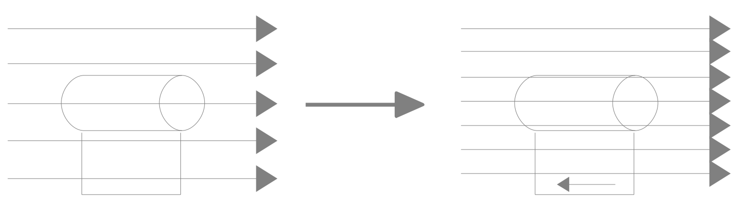
- We've already established that a moving charge or current creates a magnetic field. And as mentioned above, a charge that is moving relative to a magnetic field will feel a force.
- Combining these two ideas, we find a brand new phenomenon: If a current is flowing through a loop somewhere, and that current changes, the magnetic field it creates changes too. This changing magnetic field will affect (and move) the charges in any other loops which might be nearby.
  - Changing magnetic fields are similar to moving ones – as with electric fields, a stronger field has more densely-packed field lines. When a field weakens, for example, its lines spread out.
  - When magnetic field lines are dragged past a conductor, or when a conductor is pulled through a field, the charges inside the conductor move around – a current is produced.
- The most mathematically convenient way of explaining these induced currents is the idea of magnetic flux, which is analogous to electric flux. See the flux section in this packet for detail.
- With magnetic flux defined, it's possible to talk quantitatively about induction.
  - *Faraday's Law*: changing magnetic flux induces an emf proportional to the rate of change.
  - *Lenz's Law*: the direction of this emf is such that the induced current will "try" to prevent the change in flux.
    - The direction of the flux is not important – only the direction of the change matters. A positive flux that is increasing will induce a current that makes negative flux. A negative flux that's increasing (approaching zero, that is) will also induce a current that makes negative flux.
    - Even though flux is a scaler quantity, not a vector, it may help to "pretend" that it is a vector and think along the lines shown in the example diagram to the right.



    - Note that the magnitude of the induced flux is *less* than the change in flux. Induction tries to prevent changes in flux, but it never succeeds in preventing them. It only makes the changes slower.
  - The mathematical form of these laws: $Emf = -\frac{d\Phi_B}{dt}$

(self-)inductors

- When solenoids and current-carrying coils are introduced in physics courses, we usually only discuss them in situations where the current they carry is a constant with respect to time. However, often the current in a solenoid or coil is not constant.
- Suppose we have a solenoid resting in an externally-applied magnetic field, with a wire attached to complete a circuit, but not hooked up to any battery. If we strengthen the magnetic field, then, by Faraday's Law, the changing flux through the solenoid will induce an emf and a current will flow.



$I = 0$ $\qquad$ $I = (1/R)(d\Phi_B/dt)$

- Even if the field is produced *by the solenoid itself* because there is already a current flowing, when that field strengthens, the flux through the coils climbs, and an induced emf will present itself in an attempt to prevent the flux from increasing. Consider the following diagram, where the switch closes. As the magnetic field grows, the changing flux creates a *back emf* that partially counters the normal current of V/R.

$I = 0$ $\qquad\qquad$ $I = V/R - I_{\text{induced}} = V/R - (1/R)(\mathrm{d}\Phi_B/\mathrm{d}t)$

- This *self-inductance* means that it's "hard" to change the current flowing through a coil. Whether you want to increase it or decrease it, the change will induce an emf which works against you.
  - As with normal inductance, the inductor can never "win" and actually prevent the change. It merely slows things down.
- The intrinsic property of an inductor is its (self-)inductance. In general, this is just a measure of how much flux it creates per unit of current flowing through it. For a solenoid in particular (as opposed to a generic, arbitrary loopy wire) we can be more specific:

$$L = \frac{\Phi_B}{I} \qquad L_{\text{solenoid}} = \frac{\mu_0 N^2 A}{l}$$

- Note that here the length of the solenoid has been written as l instead of L, to avoid confusion with the symbol for inductance itself.
- The induced potential difference across an inductor can be determined using Faraday's Law and the definition of inductance given just above:

$$V_L = Emf = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(LI) = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

- If the inductor has some intrinsic resistance as well, then the total voltage drop across it will depend on both the induced voltage and the Ohmic ($V = IR$) voltage.
  - However, *usually we imagine ideal inductors that have zero resistance*, which means the induced voltage is the only voltage to worry about.
- In an LR circuit (with both resistors and inductors), the voltage across the inductor and the current through it vary exponentially, much like in an RC circuit.
  - Any time the current is adjusted, the voltage across the inductor spikes, then drops off towards zero over time. In the circuit's steady state, the voltage across the inductor will always go to zero because the current is no longer changing.
  - The current through the inductor, however, may grow or decay depending on the situation (such as which switch has been thrown and the layout of the circuit).

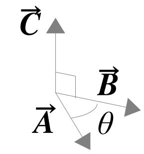
$$\text{Growth: } \frac{I}{I_0} = \left(1 - e^{(-t/\tau)}\right) \qquad \text{Decay: } \frac{V}{V_0} = \frac{I}{I_0} = e^{(-t/\tau)}$$

- The time constant for an LR circuit is $\tau = L / R$. Like the RC time constant, this τ has units of time.
- When an inductor is first brought into a circuit, you can treat it as if it has infinite R.
- When an inductor has been in a circuit for a very long time, you can treat it as if it has zero R.
- When the circuit around an inductor is changed, the inductor "wants" to maintain its old current, and will approach the new current exponentially.

OVERALL SUMMARY STUFF!

cross products and right hand rules

- Magnetism calculations often use a type of vector multiplication called a “cross product”.
- A vector “dot product” produces a scaler as a result. A “cross product” produces another vector.
- If we cross vectors $\vec{A}$ and $\vec{B}$ to get vector $\vec{C}$...
  - The direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.
  - The magnitude of $\vec{C}$ is... $|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}|\,|\vec{B}|\,\sin(\theta) = A\,B\,\sin(\theta)$
  - θ is the angle between $\vec{A}$ and $\vec{B}$.



- Given vectors $\vec{A}$ and $\vec{B}$, there are *two* directions perpendicular to both. The correct direction for $\vec{C}$ is determined using a right hand rule, and is indicated in the figure. The oppositely-pointing vector, $-\vec{C}$, would be the result of $\vec{B} \times \vec{A}$.
  - To determine this direction with your right hand, start by pointing your fingers in the direction of $\vec{A}$. Curl the fingers towards $\vec{B}$. Your thumb then points in the direction of $\vec{C}$.
- Because of the sine dependence, a cross product is largest when the input vectors are perpendicular, and the product is zero if the input vectors are parallel or antiparallel (180° apart).
- To determine the direction of a magnetic field produced by a current, there is a simpler right hand rule than using the cross-product rule and the Biot-Savart Law:
  - Imagine holding the line of current in your hand with your thumb pointing along the current. The direction that your fingers wrap around the current is the same as the direction of the magnetic field lines.
  - These two right hand rules are interchangeable – you can use whichever you prefer. If done correctly, they will always yield the same answer.

consequences of mobile charges in conductors

- A conductor is in electrical equilibrium if there is no current flowing within it. The following statements apply only to conductors *in equilibrium*.
- Any charge density is on the surface. The interior of the conductor is neutral.
  - If excess charge density were to be present inside a conductor, the charges' mutual repulsion would push them to the surface.
- The electric field in the interior is zero.
  - If a field were present in the interior, it would move the mobile charges until the secondary electric field set up by their separation exactly canceled out the initial field.
  - However, there may be a *magnetic* field in the interior, even when in equilibrium.
- The electric potential on the surface and in the interior is uniform or “flat”.
  - This is just a restatement of the electric field rule, given that $\vec{E} = -\mathrm{d}V/\mathrm{d}\vec{r}$.
- Charge density on the surface is uniform for conducting spheres, but varies for other shapes, generally concentrating in areas that stick “out” away from the rest of the object.
  - Consider two conducting spheres of radii R and r ($R > r$), connected by a very long, thin wire.
  - Since the spheres are very far apart, we can determine their electric potential by considering one sphere at a time.
  - The potential of a spherically-symmetric distribution of charge (when measured anywhere outside of that sphere's surface) is the same as if that charge were concentrated at the sphere's center, so the two spheres have potentials

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{R} \qquad v = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

    where the spheres have charges Q and q, respectively.

- Because they're electrically connected by a wire, they must be at the same electric potential. So:

$$V=v \quad \rightarrow \quad \frac{1}{4\pi\epsilon_0}\frac{Q}{R}=\frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \rightarrow \quad \frac{Q}{R}=\frac{q}{r}$$

- The spheres' surface charge densities are $\Sigma=Q/4\pi R^2$ and $\sigma=q/4\pi r^2$, so:

$$\frac{Q}{R}=\frac{q}{r} \quad \rightarrow \quad \frac{Q}{R}\left(\frac{1}{4\pi Rr}\right)=\frac{q}{r}\left(\frac{1}{4\pi Rr}\right) \quad \rightarrow \quad \frac{\Sigma}{r}=\frac{\sigma}{R} \quad \rightarrow \quad \frac{\Sigma}{\sigma}=\frac{r}{R}$$

- Hence, the charge density on the small sphere is higher. With more complicated math, this result can be extended to objects other than spheres, such as those with edges and corners. The sharper an edge or corner is (i.e., a small radius of curvature), the higher the charge density.

general properties of field lines

- Field lines are a simplified way of thinking about 3D vector fields.
  - The "real" field is a set of vectors, one for every point of space in the universe, each with its own direction and magnitude.
  - Field lines are the envelope of the field vectors.
  - In other words, the vectors are tangent to the field lines.
- Lines for a particular type of field never cross.
  - However, E field lines and B field lines may cross each other.
- Field lines *are not trajectories* for charges moving in the field.
  - There are only two cases where a charge's trajectory happens to be entirely along a field line.
    - In a uniform magnetic field, if the charge's velocity is parallel to the field lines: It will then move with constant speed along the lines, but only because it is feeling no forces.
    - In a uniform electric field, if the charge has no initial velocity perpendicular to the field lines: It will then accelerate and move along the field lines because the only force on it is along the field *and* because it has no velocity across the lines. (An example: an electron released from rest near the negative plate of a capacitor.)

flux

- Flux is a convenient mathematical way of "counting" how much field goes through a specified area or surface. Flux can be calculated for both electric and magnetic fields (and other fields too).
  - The "surface" involved in calculating flux is usually imaginary – it's an abstract mathematical construct, and does not have to be the real surface of an object.
- The simplest situation for determining flux is through a flat surface which lies in a plane perpendicular to a uniform field. There, the flux is simply the field's strength times the area.
  - However, the orientation of the area and the field lines is important. If we tilt the surface a little bit its cross sectional area drops and fewer lines go through. So, we need to throw in some trigonometry. If we define an area vector which is perpendicular to the area it represents, then using a dot product (with its intrinsic cosine) instead of normal multiplication will do the trick.
  - Also, most fields in this universe of ours are not uniform. To find the flux of a nonuniform field, it's the world's greatest mathematical trick: integration. As long as you look at a small enough patch of the surface, the field will be pretty close to uniform. So, break up the whole surface into myriad tiny parts and do the dot product for each one separately.
- With those two modifications, electric and magnetic flux are defined this way:

$$\Phi_E=\int \vec{E}\cdot \mathrm{d}\vec{A} \qquad \Phi_B=\int \vec{B}\cdot \mathrm{d}\vec{A}$$

- But, as stated above, you don't have to do an integral every time. If the field has the same strength at every point on your surface, or even if you can break the surface up into a few pieces you can save yourself the trouble of a whole integral. For example:

$$\text{Uniform field: } \Phi_{\mathrm{B}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}} \qquad \text{Two patches: } \Phi_{\mathrm{B}} = \vec{\boldsymbol{B}}_1 \cdot \vec{\boldsymbol{A}}_1 + \vec{\boldsymbol{B}}_2 \cdot \vec{\boldsymbol{A}}_2$$

- Note that because of the dot product, the flux of a field which is parallel to your surface is zero. This is a very useful trick to keep in mind when you are picking a surface to integrate over!

things you might need to integrate

- Charges
  - Done to find $\vec{\boldsymbol{F}}$, $\vec{\boldsymbol{E}}$, V, or U.
  - Integrate all dq elements, which may be in a line, area, or volume.
- Currents
  - Done to find $\vec{\boldsymbol{B}}$ using Biot-Savart Law.
  - Integrate all $d\vec{\boldsymbol{s}}$ elements crossed with $\vec{\boldsymbol{r}}$, where $d\vec{\boldsymbol{s}}$ is an element of the current's path.
- Paths
  - Done to find $\vec{\boldsymbol{B}}$ or I using Ampere's Law.
  - Integrate all $d\vec{\boldsymbol{s}}$ elements dotted with $\vec{\boldsymbol{B}}$, where $d\vec{\boldsymbol{s}}$ is an element of an imaginary closed-loop path. The loop is usually floating in empty space, and is not an actual loop of wire. There is no current flowing along this path – rather, the current I passes through or penetrates it.
- Areas
  - Done to find electric flux for Gauss' Law, or to find magnetic flux for Faraday's Law.
  - For Gauss, integrate all $d\vec{\boldsymbol{a}}$ elements dotted with $\vec{\boldsymbol{E}}$, where $d\vec{\boldsymbol{a}}$ is an element of an imaginary closed surface. The surface is usually floating in empty space, and is not the real surface of an object. There are usually not any charges *on* it, only inside it.
  - For Faraday, integrate all $d\vec{\boldsymbol{a}}$ elements dotted with $\vec{\boldsymbol{B}}$, where $d\vec{\boldsymbol{a}}$ is an element of an imaginary surface. The surface must be bound by a closed loop through which you want to find the flux. This imaginary surface also floats in empty space. The loop that binds it is often a real wire, but it doesn't have to be. There can be an emf floating in space with no wire to carry current.

Summary by Steve Stonebraker, last revised September 2015.